equation with shear /9/, which has a closed solution.

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HIGH-FREQUENCY SHEAR OSCILLATIONS OF A STRIP STAMP ON AN ELASTIC HALF-SPACE*

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Harmonic shear oscillations of a rigid stamp with a plane base coupled to an elastic half-space are studied. The problem is reduced to dual integral equations whose solution, effective for short waves, was constructed in /1/ and validated in /2/. Expressions for the complex amplitude of stamp oscillations are given and analysed, as well as those for the contact stresses and their intensity coefficients, and the power averaged over the period of oscillations transmitted from the stamp to the half-space per unit length of the stamp.

The problems of the oscillation of a strip stamp on an elastic half-space have been the subject of a considerable number of publications (see /3, 4/ and their bibliography). However, a high-frequency analysis of the oscillation patterns is practically non-existent. The only known reference on the subject in a short section in /5/.

Consider a rigid strip stamp whose plane foot occupies a strip $|x| < a,\ y = 0,\ |z| < \infty$. The stamp is coupled to a homogeneous isotropic elastic half-space $y > 0$ and executes oscillations along the Oz axis under the action of a load harmonically time-dependent, with linear density $\operatorname{Re}(Te^{i\omega t})$. We will write the unique non-zero displacement component in the form

$$u_z = u_z(x, y, t) = \operatorname{Re}[w(x, y)e^{i\omega t}]$$

$$\Delta w + k^2 w = 0,\ y > 0;\ k = \omega c^{-1},\ c = G^{1/2}\rho^{-1/2} \tag{1}$$

$$w|_{y=0} = \delta,\quad |x| < a;\quad \left.\frac{\partial w}{\partial y}\right|_{y=0} = 0,\quad |x| > a \tag{2}$$

The function $w(x, y)$ satisfies the demand of local finiteness of the energy and radiation conditions; G is the shear modulus, δ is the unknown complex amplitude of the stamp oscillations and c is the velocity of volume shear waves. The problem is closed by the equation of motion of the stamp in complex form

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$$T + \int_{-a}^{a} \tau(x)\, dx = -m\omega^2\delta,\ \ \tau(x) = \tau_{yz}(x, 0) = G \frac{\partial w}{\partial y}\Big|_{y=0} \tag{3}$$

where m is the running mass of the stamp and $\tau(x)$ is the contact stress.

Taking (1), the radiation conditions and the symmetry of the problem into account, we seek $w(x, y)$ in the form

$$w(x, y) = \delta \int_0^\infty M(\nu)\, e^{-y\sqrt{\nu^2-k^2}} \cos \nu x\, d\nu \tag{4}$$

Then condition (2) leads to dual integral equations for $M(\nu)$

$$\int_0^\infty M(\nu) \cos \nu x\, d\nu = 1, \quad 0 \leqslant x < a \tag{5}$$
$$\int_0^\infty \sqrt{\nu^2 - k^2}\, M(\nu) \cos \nu x\, d\nu = 0, \quad x > a$$

A solution of these equations suitable for short waves $ka \gg 1$ was constructed in /1/ and validated in /2/. Introducing the notation

$$p_\pm(\nu, t) = \frac{e^{i\nu t}}{\sqrt{i(k+\nu)}} \pm \frac{e^{-i\nu t}}{\sqrt{i(k-\nu)}}$$
$$q_\pm(\nu, t) = \frac{e^{i\nu t}}{\sqrt{i(k-\nu)}} \pm \frac{e^{-i\nu t}}{\sqrt{i(k+\nu)}}$$

we find, that according to /1/

$$M(\nu) = \frac{\sqrt{k}}{\pi i\nu} q_-(\nu, a) - N(\nu), \quad N(\nu) = \frac{\sqrt{ki}}{\pi} \int_a^\infty \psi(t)\, q_+(\nu, t)\, dt \tag{6}$$

and the function $\psi(t)$ satisfies the integral equation of the second kind

$$\psi(t) = -\frac{k}{2}\left[\int_a^\infty \Lambda(k(t+s))\, ds + \int_a^\infty \psi(s)\, \Lambda(k(t+s))\, ds\right], \tag{7}$$
$$a \leqslant t < \infty$$
$$\Lambda(\xi) = H_0^{(2)}(\xi) - iH_1^{(2)}(\xi)$$

($H_\alpha^{(2)}(\xi)$ is the Henkel function of the second kind). For sufficiently large values of ka the operator appearing in (7) is a compression operator in a suitable functional space in which a solution of (7) exists, is unique /2/ and has the following asymptotic expansion /1/:

$$\psi(t) = \frac{e^{-ik(t-a)}}{\sqrt{\pi}\,(2ik(t+a))^{3/2}}\left[1 - \frac{15}{8ik(t+a)} + \ldots\right] \tag{8}$$

Thus to obtain a complete solution it remains to determine the amplitude δ. With this in mind, we find the contact stress

$$\tau(x) = G \frac{\partial w}{\partial y}\Big|_{y=0} = -G\delta \int_0^\infty M(\nu) \sqrt{\nu^2-k^2} \cos \nu x\, d\nu, \quad |x| < a$$

Using Eqs.(6), the identity

$$N(\nu)\sqrt{\nu^2-k^2} = -\frac{\sqrt{ik}}{\pi}\left\{\psi(a)\, p_+(\nu, a) + \int_a^\infty e^{ikt}\, [\psi(t)\, e^{-ikt}]'\, p_+(\nu, t)\, dt\right\}$$

arising from it and the values of the discontinuous integrals

$$\frac{1}{\sqrt{\pi}} \int_0^\infty p_+(\nu, t) \cos \nu x\, d\nu = \frac{e^{-ik(t+x)}}{\sqrt{t+x}} + h(t-x) \frac{e^{-ik(t-x)}}{\sqrt{t-x}} \tag{9}$$
$$\frac{\sqrt{k}}{\pi} \int_0^\infty p_-(\nu, t) \frac{\cos \nu x}{\nu}\, d\nu = -1 + \operatorname{erf}(\sqrt{ik(t+x)}) + h(t-x) \operatorname{erf}(\sqrt{ik(t-x)})$$

($h(\xi)$ is the unit Heaviside function), we find the following expression for the contact stress

$$\tau(x) = -\delta G \sqrt{\frac{ik}{\pi}}\left\{[1+\psi(a)]\left[\frac{e^{-ik(a-x)}}{\sqrt{a-x}} + \frac{e^{-ik(a+x)}}{\sqrt{a+x}}\right] + \int_a^\infty e^{ikt}\, [\psi(t)\, e^{-ikt}]' \left[\frac{e^{-ik(t-x)}}{\sqrt{t-x}} + \frac{e^{-ik(t-x)}}{\sqrt{t+x}}\right] dt + \right. \tag{10}$$
$$\left. \sqrt{ik\pi}\left[\operatorname{erf}(\sqrt{ik(a-x)}) + \operatorname{erf}(\sqrt{ik(a+x)}) - 1\right]\right\}, \quad |x| < a$$

From (10) we see that the condition of finiteness of the energy is satisfied near the stamp edges, and the following formula forthe stress intensity coefficient is obtained:

$$K_{III} = \lim_{x \to a-0} \sqrt{2\pi(a-x)}\,\tau(x) = -\delta G \sqrt{2ik}\,[1 + \psi(a)] \tag{11}$$

Integrating Eq.(10) with respect to x, we obtain the running reaction of the foundation in the form

$$\int_{-a}^{a} \tau(x)\,dx = -\delta G Q(ka) \tag{12}$$

$$Q(ka) = -2ika + (4ika - 1)\operatorname{erf}(\sqrt{2ika}) + 2\sqrt{\frac{2ika}{\pi}}\,e^{-2ika} + ik\int_{a}^{\infty} \psi(t)\,[\chi_-(t) - \chi_+(t)]\,dt, \quad \chi_{\pm}(t) = \operatorname{erf}(\sqrt{ik(t \pm a)}) + \frac{e^{ik(t\pm a)}}{\sqrt{\pi ik(t \pm a)}}$$

$$Q(ka) = 2ika + 1 - e^{-2ika}\,[2\pi^{-1/2}(2ika)^{1/2} + O((ka)^{-3/2})], \quad ka \to \infty \tag{13}$$

The equation of motion of the stamp (3) can be represented, by virtue of (12), in the form (μ is the stamp inertia coefficient

$$\delta\,[\mu(ka)^2 - Q(ka)] = -TG^{-1}, \quad \mu = m/(\rho a^2) \tag{14}$$

Equations (12)–(14) completely define the quantity δ and enable us to obtain its dynamic expansion in powers of the small parameter $(ka)^{-1/2}$. The specific form of this expansion depends on whether the stamp is massive, or not. If the quantity $m \neq 0$ is fixed, then we have, within the specified accuracy (henceforth we introduce the notation $O_j = O((ka)^{-j/2})$)

$$\delta = -T_*\left(1 + \frac{2i}{\mu ka} + O_3\right), \quad T_* = \frac{T}{G\mu(ka)^2}, \quad \mu \neq 0 \tag{15}$$

$$|\delta| = T_*(1 + O_4), \ \arg\delta = \pi + O_2, \quad \operatorname{Im}\delta = -2T_*(\mu ka)^{-1}(1 + O_1)$$

as $ka \to \infty$, otherwise ($m = 0$) we have

$$\delta = -iT_*\left(1 + \sqrt{\frac{2}{\pi ika}}\,e^{-2ika} + O_2\right) \tag{16}$$

$$T_* = \frac{T}{2Gka}, \quad |\delta| = T_*(1 - O_1)$$

$$\arg\delta = -\pi/2 + O_1, \ \operatorname{Im}\delta = -T_*(1 + O_1), \ \mu = 0$$

The solution of the problem obtained enables us to write a high-frequency expansion for any quantity appearing in the problem, which is interesting from the practical point of view. Thus according to (11), (8), (15), (16) we have the following expression for the stress intensity coefficient:

$$K_{III} = \frac{T}{\mu}\sqrt{\frac{2i}{a}}\,(ka)^{-3/2}(1 + O_2), \quad \arg K_{III} = \frac{\pi}{4} + O_2, \quad m \neq 0$$

$$K_{III} = -\frac{T}{\sqrt{a}}\,(2ika)^{-1/2}(1 + O_1), \quad \arg K_{III} = \frac{3\pi}{4} + O_1, \quad m = 0$$

Let us determine the power W_0, averaged over a period of the oscillations, transmitted by the stamp to the half-space per unit length of the stamp. Calculating the mean power of the force $\operatorname{Re}(Te^{i\omega t})$ during the stamp displacement $\operatorname{Re}(\delta e^{i\omega t})$, we obtain $W_0 = -{}^1/_2\,\omega T\ \operatorname{Im}\delta$, and from (15), (16) we have, in the high-frequency case,

$$\frac{W_0}{cT} = \frac{T}{\mu^2 Ga}(ka)^{-2}(1 + O_1), \quad m \neq 0$$

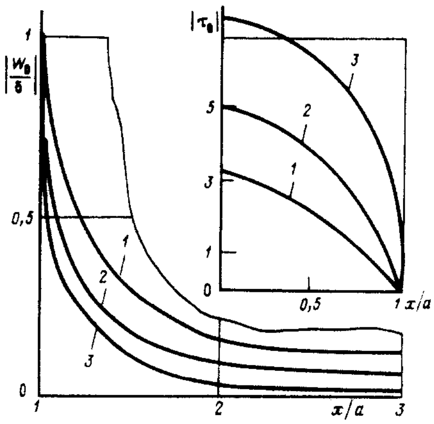
$$\frac{W_0}{cT} = \frac{T}{4Ga}(1 + O_1), \quad m = 0$$

These formulas show that if the stamp is massive ($\mu > 0$), the energy transmitted by the stamp to the half-space over a period, per unit length of the stamp is proportional, at high frequencies, to $\mu^{-2}\omega^{-3}$; (if on the other hand the stamp is inertialess ($\mu = 0$), this quantity is much greater and proportional to ω^{-1}.

Let us turn our attention to calculating the displacement of the points lying on the surface of the half-space. Subsituting expressions (6) into (4) at $y = 0$, taking into account the second formula of (9) and the following expression:

$$\frac{1}{\sqrt{\pi}}\int_0^\infty q_+(\nu, t)\cos \nu x\, d\nu = h(|x|-t)\frac{e^{ik(|x|-t)}}{\sqrt{|x|-t}}, \quad t>0$$

and using expression (8) for $\psi(t)$, we arrive at the following asymptotic formula uniformly suitable for $|x|\geqslant a$



$$w_0(x)=\delta\Big\{1-\mathrm{erf}\left(\sqrt{ik(|x|-a)}\right)- \frac{e^{ik(|x|-a)}}{2ika\sqrt{\pi}}\left(\frac{|x|}{a}-1\right)^{1/2}\left[\left(\frac{|x|}{a}+1\right)^{-1}- \frac{5}{16ika}\left(\frac{|x|}{a}+5\right)\left(\frac{|x|}{a}+1\right)^{-2}+O_4\right]\Big\}, \quad |x|\geqslant a \qquad (17)$$

The dependence of $|w_0(x)/\delta|$ on x/a is shown in the figure for the values $ka = 4\pi, 8\pi, 20\pi$ (curves *1–3* respectively). The manner in which $|w_0(x)/\delta|$ decreases as x/a increases for fixed ka is well shown; for a given value of x/a the quantity $|w_0(x)/\delta|$ decreases monotonically as ka increases.

Let us now find a high-frequency asymptotic expression suitable for $|x|\leqslant a$, from the formulas (10), (8). The uniform asymptotic expansion of the integral term in (10) is constructed using the results of /6/ (see also /7/), and its contribution is of order $O_2 = O((ka)^{-1})$ for all $|x|<a$. Thus the asymptotic formula for the stress takes the form

$$\tau(x) = -\frac{\delta G}{a}\sqrt{\frac{ika}{\pi}}\Big\{\sqrt{\pi ika}\left[\mathrm{erf}\left(\sqrt{ik(a-x)}\right)+ \mathrm{erf}\left(\sqrt{ik(a+x)}\right)-1\right]+O_2+\left[\left(1-\frac{x}{a}\right)^{-1/2}e^{-ik(a-x)}+ \left(1+\frac{x}{a}\right)^{-1/2}e^{-ik(a+x)}\right](1+O_3), \quad |x|<a$$

The distribution of the dimensionless quantity $|\tau_0(x)|$ where

$$\tau_0(x) = \frac{\tau(x)}{Gk\delta}\left[ka\left(1-\frac{x^2}{a^2}\right)\right]^{1/2}$$

is shown in the figure (the curves are numbered exactly as in the case of the displacement graphs). It is clear that the non-uniformity of the distribution $|\tau_0(x)|$ over the stamp foot increases as the frequency increases.

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